Jacobi Elliptic Functions

Theory and Applications

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Contents

INTRODUCTION

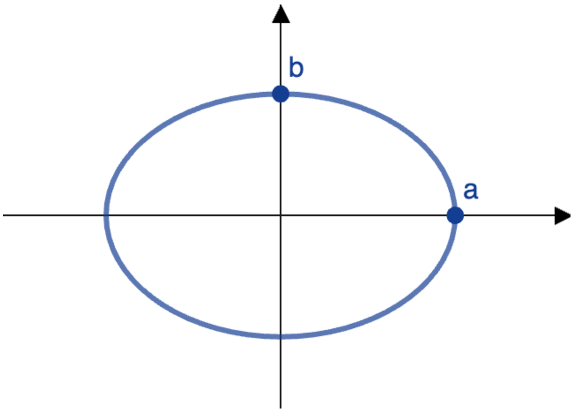
Jacobi elliptic functions arise from a mathematical frustration: the inability of elementary functions (polynomials, rational fractions, trigonometric functions, etc.) to express certain fundamental measurements.

The initial problem, which gives these functions their name, is the calculation of the arc length of an ellipse. Unlike the circle, whose circumference is easy to compute, the ellipse requires the use of so-called elliptic integrals, which have no antiderivatives expressible in terms of classical functions.

To resolve this obstacle, Jacobi had the idea of defining new functions by inversion of these integrals. In doing so, he discovered a mathematically rich universe.

I - DEFINITIONS AND GENERAL PROPERTIES

1 - Ellipses



The study of Jacobi functions rests on the geometry of the ellipse.

Definition (D1): Equation of the ellipse

Let $a, b \in \mathbb{R}$.

The ellipse centred at the origin of the Cartesian plane, with semi-major axis a and semi-minor axis b, is defined by the equation:

$$\forall x, y \in \mathbb{R}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Definition (D2): Eccentricity

Let $a, b \in \mathbb{R}$ such that $a \geq b > 0$.

The ellipse is characterised by a fundamental parameter: its eccentricity, denoted k, which measures its flatness. It lies in the interval $[0, 1[$ and is defined by the relation:

$$k = \sqrt{1 - \frac{b^2}{a^2}}.$$

Remark:

In the context of elliptic functions, k is called the modulus.

For $k = 0$ (i.e. $a = b$), the ellipse is a circle.

As k tends towards 1, the ellipse is stretched until it tends towards a line.

Remark for what follows:

We place ourselves in the case where $b = 1$ and $a > 1$, as this is the framework within which elliptic functions can be considered.

2 - Definition

Throughout this section, let $k \in [0, 1[$ be fixed.

a - Elliptic integral of the first kind

To pass from the geometry of the ellipse to the Jacobi functions, it is necessary to introduce the elliptic integral.

Definition (D3): Elliptic integral

Let $\alpha \in \mathbb{R}$.

The incomplete elliptic integral of the first kind, denoted $F(\alpha, k)$, is defined by:

$$F(\alpha, k) = \int_0^\alpha \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}.$$

α is called the amplitude of the integral.

b - Definition by inversion

The fundamental concept of Jacobi lies in the inversion of the elliptic integral.

Definition (D4): Jacobi amplitude

Let $u \in \mathbb{R}$.

We define the Jacobi amplitude, denoted $am(u,k)$, as the inverse function of the elliptic integral such that:

$$u = F(\alpha, k) \iff \alpha = am(u,k).$$

Definition (D5): Elliptic functions

Let $u \in \mathbb{R}$.

From the function $am(u,k)$, we define the three primary elliptic functions:

Jacobi sine: $sn(u,k) = \sin(am(u,k))$,

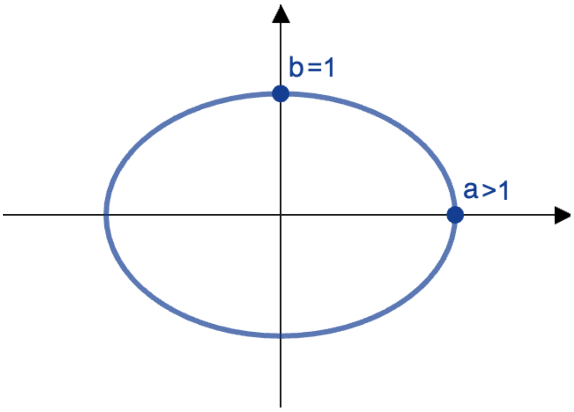
Jacobi cosine: $cn(u,k) = \cos(am(u,k))$,

Jacobi delta function: $dn(u,k) = \sqrt{1 - k^2 sn^2(u,k)}$.

Remark:

Unlike classical trigonometric functions, which depend only on one variable, these functions take two variables: they depend on both the variable u and the modulus k.

c - Real graphical interpretation



We recall that we are in the specific case where $b = 1$ and $a > 1$. In this setting, the Jacobi functions admit a direct geometric interpretation linked to the coordinates of a point on the ellipse.

Proposition (P1): Position on the ellipse

Let $u \in \mathbb{R}$,

Let $(x, y) \in \mathbb{R}^2$.

Consider the point with coordinates (x,y) moving along the ellipse of modulus k.

Let $r := \sqrt{x^2 + y^2}$ denote its distance from the origin. Then its position can be parametrised as follows:

Ordinate: $y = sn(u, k)$,

Normalised abscissa: $\frac{x}{a} = cn(u, k)$,

Normalised radius vector: $\frac{r}{a} = dn(u, k)$.

Proof. (P1): Let $\alpha := am(u, k)$ be the amplitude of the motion.
From the equation of the ellipse, for $b = 1$, we have:
$\frac{x^2}{a^2} + y^2 = 1$.
By analogy with the equation of the circle, we have $y = \sin(\alpha)$.
Since $sn(u, k) = \sin(\alpha)$, we get $y = sn(u, k)$.
Substituting into the ellipse equation, we obtain:
$\frac{x^2}{a^2} + \sin^2(\alpha) = 1$,
so $\frac{x^2}{a^2} = 1 - \sin^2(\alpha)$,
so $\frac{x^2}{a^2} = \cos^2(\alpha)$,
so $\frac{x}{a} = \cos(\alpha)$ (taking the positive root),
so $\frac{x}{a} = cn(u, k)$ from the above.
We know that $r = \sqrt{x^2 + y^2}$.
We have $(\frac{r}{a})^2 = \frac{r^2}{a^2} = \frac{x^2+y^2}{a^2} = \frac{x^2}{a^2} + \frac{y^2}{a^2}$,
so $(\frac{r}{a})^2 = cn^2(u, k) + \frac{sn^2(u,k)}{a^2}$,
so $(\frac{r}{a})^2 = 1 - sn^2(\alpha) + \frac{sn^2(\alpha)}{a^2}$,
so $(\frac{r}{a})^2 = 1 - sn^2(\alpha)(1 - \frac{1}{a^2})$.
Since $k = \sqrt{1 - \frac{b^2}{a^2}}$ we obtain $k^2 = 1 - \frac{1}{a^2}$.
Thus $(\frac{r}{a})^2 = 1 - k^2 sn^2(\alpha)$.
We will show later that $dn^2(u) + k \cdot sn^2(u) = 1$ (P8).
We therefore obtain $(\frac{r}{a})^2 = dn^2(u)$.
Hence $\frac{r}{a} = dn(u, k)$. ■

Proposition (P2): Differential relation

Let $\alpha \in \mathbb{R}$,

Let $\theta \in [0, \alpha]$,

Let r be defined as above.

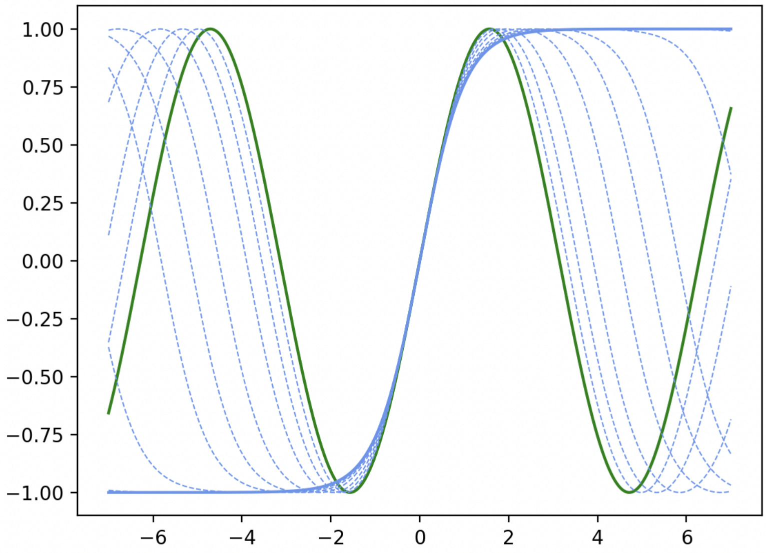
Then: $du = \frac{a}{r} d\alpha$.

Proof. (P2): We know that $u = \int_0^\alpha \frac{d\theta}{\sqrt{1-k^2 \sin^2(\theta)}} (*)$.
By the fundamental theorem of calculus, differentiating gives:
$\frac{du}{d\alpha} = \frac{1}{\sqrt{1-k^2 \sin^2(\alpha)}}$,
so $du = \frac{d\alpha}{\sqrt{1-k^2 \sin^2(\alpha)}}$.
We know that $dn(u, k) = \sqrt{1 - k^2 sn^2(u, k)}$ and that $sn(u, k) = \sin(\alpha)$,
so $dn(u, k) = \sqrt{1 - k^2 \sin^2(\alpha)}$.
Moreover, $dn(u, k) = \frac{r}{a}$,
so $\sqrt{1 - k^2 \sin^2(\alpha)} = \frac{r}{a}$.
Substituting into (*), we obtain $du = \frac{a}{r} d\alpha$. ∎

Remark:

Thus, while the circular trigonometric functions describe motion on a circle ($a = b = 1$), the Jacobi functions describe motion and distances along an elliptical trajectory.

Visual interpretation:

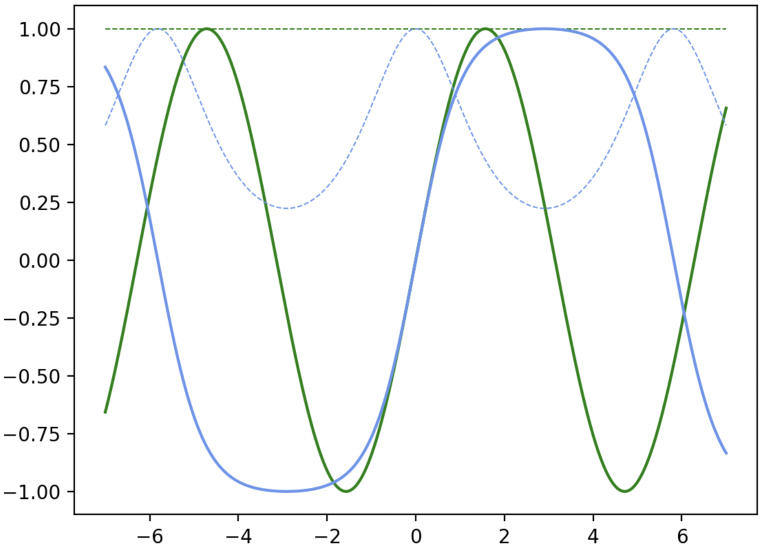


The curves of these functions can be plotted in the real plane.

Above is the function sn, varying u for different values of k. The green curve represents $sn(u, k)$ for k=0.

It corresponds to the sine function. As k increases, the sinusoid deforms and stretches.

In the limit, as $k \to 1$, the function corresponds to tanh.



The function dn allows one to visualise the deformation of the sinusoid.

The solid green and blue curves represent $sn(u, 0)$ and $sn(u, 0.7)$ respectively.

The more the sn function is stretched, the more pronounced the variations in dn.

3 - Extension to the complex plane

To understand the deep nature of the Jacobi functions, it is necessary to extend their study to the complex plane.

This approach allows them to be defined no longer merely by an integral calculation, but by their structural properties.

We again fix $k \in [0, 1[$.

a - Notions on complex functions

Let $D \subset \mathbb{C}$

Definition (D6): Holomorphic function

A function is said to be holomorphic on D if it is differentiable in the complex sense at every point of D. The set of holomorphic functions from D to C is denoted $\mathcal{H}(D)$.

Remark:

These functions are extremely regular and can be expanded as power series around any point of their domain.
They are said to be analytic.

Definition (D7): Meromorphic function

A function is said to be meromorphic if it can be defined as the quotient of two holomorphic functions.
The set of meromorphic functions from D to C is denoted $\mathcal{M}(D)$.

Remark:

A meromorphic function is therefore "holomorphic almost everywhere", except at certain isolated points where the denominator vanishes.
We thus have $\mathcal{H}(D) \subset \mathcal{M}(D)$.

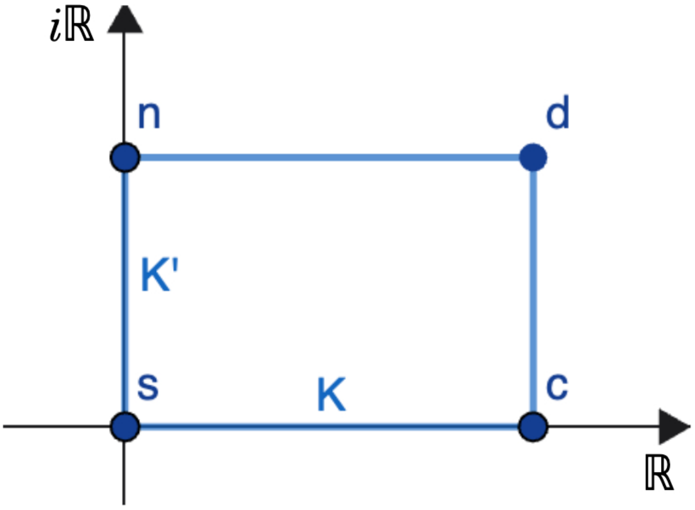
Definition (D8): Zero and Pole

Let $g, h \in \mathcal{H}(D)$,
Let $f \in \mathcal{M}(D)$ such that $f = \frac{g}{h}$,
Let $p, q \in D$,
The behaviour of a meromorphic function is characterised by its singular points:
Zero: If $g(p) = 0$ (and $h(p) \neq 0$), we say that p is a zero of f.
At such a point, $f(p) = 0$.
Pole: If $h(q) = 0$ (and $g(q) \neq 0$) we say that q is a pole of f.
At such a point, $\lim_{z \to q} |f(z)| = +\infty$.

Remark:

The Jacobi functions (denoted pq) are particular meromorphic functions: they are constructed so as to possess a unique simple zero at vertex p and a unique simple pole at vertex q within a fundamental rectangle called the Jacobi rectangle.

b - Jacobi rectangle



Definition (D9): Quarter-periods

The rectangle is defined by two parameters called "quarter-periods" which depend on the modulus k of the ellipse:

real period: $K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$,

imaginary period: $K' = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2(\theta)}}$.

Definition (D10): Vertices of the rectangle

We define four points, called vertices, in the complex plane:

s (source): placed at the origin 0,

c (corner): placed at distance K on the real axis,

n (nadir): placed at distance K' on the imaginary axis,

d (diagonal): placed at the point $K + iK'$.

Proposition (P3): Construction of the functions

A Jacobi function pq is the unique meromorphic function satisfying:

Zero and Pole:

It has a zero at p and a simple pole at q.

Asymptotic behaviour near 0:

If $p = s$ (zero at the origin), then $pq(u,k) \sim u$ as $u \to 0$,

if $q = s$ (pole at the origin), then $pq(u,k) \sim \frac{1}{u}$ as $u \to 0$,

otherwise (s is an ordinary point), then $pq(0,k) = 1$.

Proof. (P3): Admitted; the uniqueness of this construction rests on Liouville's theorem.

■

Examples:

$sn(u,k)$: Zero at s (0), pole at n (iK'). At the origin, $sn(u) \sim u$

$cn(u,k)$: Zero at c (K), pole at n (iK'). At the origin, $cn(0) = 1$

$dn(u,k)$: Zero at d $(K + iK')$, pole at n (iK'). At the origin, $dn(0) = 1$

Remark:

Beyond the fundamental trio (sn, cn, dn), there are in total 12 Jacobi elliptic functions, the other 9 being: cd, cs, ds, dc, ns, nc, nd, sc and sd.

They correspond to all possible combinations of ratios between the four vertices of the rectangle (s,c,d,n).

In practice, only sn, cn and dn are commonly used.

The notation is extremely intuitive as the following properties hold.

Proposition (P4): Reciprocal rule

Let $p, q \in \{s, c, d, n\}$ be two distinct vertices.

Then: $pq = \frac{1}{qp}$.

Proof. (P4): By definition, pq has a zero at p and a pole at q.

The inverse function swaps these roles: the zero of qp (at q) becomes a pole for $\frac{1}{qp}$, and the pole of qp (at p) becomes a zero for $\frac{1}{qp}$.

Both functions having the same zeros, the same poles and the same asymptotic behaviour, they are identical by uniqueness.

Hence the result. ■

Proposition (P5): Quotient rule

Let $p, q \in \{s, c, d, n\}$ be two distinct vertices.

Then: $pq = \frac{pn}{qn}$.

Proof. (P5): The numerator pn has a zero at p and a pole at n.

The denominator qn has a zero at q and a pole at n.

In the ratio $\frac{pn}{qn}$ the poles at n cancel.

What remains is a zero at p and a pole at q (since the zero of qn in the denominator becomes a pole of the quotient).

This is exactly the definition of the function pq.

Hence the result by uniqueness. ■

Examples:

$ns(u, k) = \frac{1}{sn(u,k)}$

$cs(u, k) = \frac{cn(u,k)}{sn(u,k)}$

c - Complex graphical interpretation

Visual interpretation:

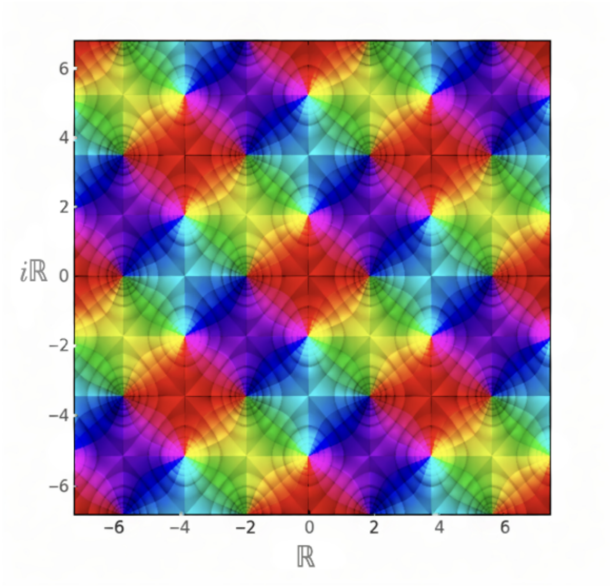
To visualise a complex function, domain colouring is often used.

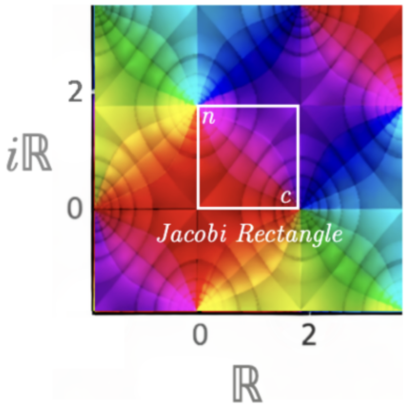
In this representation, each point of the complex plane is coloured according to the value the function takes at that point: the colour represents the argument, and the brightness represents the modulus.

In the illustration above, the function cn is shown in the complex plane. With modulus $k = 0.8$, the dimensions of the Jacobi rectangle are approximately 2 on the real axis (K) and 1.75 on the imaginary axis (K').

Zeros (Vertices c):

On the real axis, the black points where all colours converge are the simple zeros. The first is located at point K, with $K \simeq 2$. The total darkness indicates that the modulus of the function is zero there.





Poles (Vertices n):
The extremely bright and luminous centres of radiance are the simple poles. The first is located at the point iK', with $K' \simeq 1.75$. The luminous saturation confirms that the modulus of the function tends to infinity.

The periodic structure:
The pattern repeats identically every 4K horizontally and every 4iK' vertically. This double periodicity of the function cn is clearly visible.

II - FORMULAE AND PROPERTIES

Unless otherwise stated, throughout this section we fix $k \in [0, 1[$

Definition (D11): Domain of definition

We define K and K' as in the previous section.
The Jacobi functions being meromorphic, we define their domain of validity U as the open subset of C deprived of the set of poles Λ such that:
$U := \mathbb{C} \setminus \Lambda$,
where $\Lambda = \{2mK + i(2n+1)K' | (m, n) \in \mathbb{Z}^2\}$.

Remark:
On this set U, the functions sn, cn and dn are holomorphic, which guarantees the validity of the relations presented below.

Notation:

For clarity, we will use the abbreviated notations $sn(u)$, $cn(u)$, $dn(u)$ and $am(u)$ to denote $sn(u,k)$, $cn(u,k)$, $dn(u,k)$ and $am(u,k)$ respectively.

Preliminary remark:

There are, of course, other formulae beyond those proved in this section. The analogy with trigonometry being very strong, one could find square transformation formulae, half-angle formulae, etc.

1 - Special values

Proposition (P6): Special values

We have, at the points 0 and K:

$sn(0) = 0$, $sn(K) = 1$,

$cn(0) = 1$, $cn(K) = 0$,

$dn(0) = 1$, $dn(K) = \sqrt{1-k^2}$.

Proof. (P7): Follows from P9 and P10; see below. ■

Remark:

The expression $\sqrt{1-k^2}$ is sometimes called the "co-modulus" and is denoted k'.

2 - Fundamental relations

Proposition (P7): Trigonometric invariant

$\forall u \in U$, $sn^2(u) + cn^2(u) = 1$

Proof. (P7): Let $u \in U$,

Set $\alpha := am(u)$.

We have $sn(u) = \sin(\alpha)$ and $cn(u) = \cos(\alpha)$.

We know that $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

Therefore $sn^2(u) + cn^2(u) = 1$.

Hence the result. ■

Proposition (P8): Elliptic invariant

$\forall u \in U$, $dn^2(u) + k^2 \cdot sn^2(u) = 1$

Proof. (P8): Let $u \in U$,
We know that $dn(u) = \sqrt{1 - k^2 sn^2(u,k)}$.
So $dn^2(u) = 1 - k^2 sn^2(u,k)$,
so $dn^2(u) + k^2 sn^2(u,k) = 1$.
Hence the result. ■

3 - Interpolation

Proposition (P9): Circular regime

For all $u \in U$:
$sn(u,0) = \sin(u)$,
$cn(u,0) = \cos(u)$,
$dn(u,0) = 1$.

Proof. (P9): We know that $u = \int_0^\alpha \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$.
For $k = 0$, we have $u = \int_0^\alpha d\theta = \alpha$,
so $am(u,0) = u$,
so $sn(u,0) = \sin(u)$,
and $cn(u,0) = \cos(u)$,
and $dn(u,0) = 1$ ■

Proposition (P10): Hyperbolic regime

For all $u \in U$:
$sn(u,1) = \tanh(u)$,
$cn(u,1) = \mathrm{sech}(u) = \frac{1}{\cosh(u)}$,
$dn(u,1) = \mathrm{sech}(u) = \frac{1}{\cosh(u)}$.

Proof. (P10): Admitted; same principle as P9 but with more technical integral calculations. ■

Remark:

When the modulus is zero, the ellipse becomes the unit circle. The Jacobi functions reduce to the classical trigonometric functions.

When the modulus tends to 1, the ellipse stretches to infinity and becomes a line.

The functions lose their real periodicity, which is why 1 is excluded from the domain of definition of k.

4 - Period

Proposition (P11): Periodicity

For all $u \in U$:

$sn(u + 4K) = sn(u)$,

$cn(u + 4K) = cn(u)$,

$dn(u + 2K) = dn(u)$.

Proof. (P11): Let $u \in U$.

For $u = K$ we have $am(u) = \frac{\pi}{2}$,

so for $u = 2K$ we have $am(u) = \pi$,

and for $u = 4K$ we have $am(u) = 2\pi$.

We have $sn(u + 4K) = \sin(am(u) + 2\pi) = \sin(am(u)) = sn(u)$ by 2π-periodicity of the sine function.

Likewise, $cn(u + 4K) = cn(u)$.

Similarly for dn using the π-periodicity of $\sin^2$.

Hence the result. ■

Remark:

This explains why K is called the “quarter-period”.

5 - Parity

Lemma (L1): Odd parity of the amplitude

$\forall u \in U$, $am(-u) = -am(u)$

Proof. (L1): Admitted. ■

Proposition (P12): Parity

For all $u \in U$:

$sn(-u) = -sn(u)$,

$cn(-u) = cn(u)$,

$dn(-u) = dn(u)$.

Proof. (P12): Let $u \in U$.

We have $sn(-u) = \sin(am(-u)) = \sin(-am(u)) = -\sin(am(u)) = -sn(u)$ by the lemma and the odd parity of sine.

Likewise for cn.

We know that $dn^2(u) + k^2 \cdot sn^2(u) = 1$,

so $dn^2(u) = 1 - k^2 \cdot sn^2(u)$,

so $dn^2(-u) = 1 - k^2 \cdot sn^2(-u)$.

But $sn^2(-u) = (-sn(u))^2 = sn^2(u)$.

So $dn^2(-u) = 1 - k^2 \cdot sn^2(u)$,

so $dn^2(-u) = dn^2(u)$,

so $dn(-u) = dn(u)$.

Hence the result. ■

6 - Addition and duplication

Proposition (P13): Addition formulae

For all $(u, v) \in U^2$ such that $(u + v) \in U$, we have:

$sn(u+v) = \frac{sn(u)cn(v)dn(v)+sn(v)cn(u)dn(u)}{1-k^2sn^2(u)sn^2(v)}$,

$cn(u+v) = \frac{cn(u)cn(v)-sn(u)sn(v)dn(u)dn(v)}{1-k^2sn^2(u)sn^2(v)}$,

$dn(u+v) = \frac{dn(u)dn(v)-k^2sn(u)sn(v)cn(u)cn(v)}{1-k^2sn^2(u)sn^2(v)}$.

Proof. (P13): Admitted. ■

Proposition (P14): Duplication formulae

For all $u \in \mathbb{C}$:
$sn(2u) = \frac{2sn(u)cn(u)dn(u)}{1-k^2sn^4(u)}$,
$cn(2u) = \frac{cn^2(u)-sn^2(u)dn^2(u)}{1-k^2sn^4(u)}$,
$dn(2u) = \frac{dn^2(u)-k^2sn^2(u)cn^2(u)}{1-k^2sn^4(u)}$.

Proof. (P14): Immediate from P13 by considering "$sn(u+u)$", "$cn(u+u)$" and "$dn(u+u)$".
■

7 - Differential and integral calculus

Remark:

The differentiation and integration formulae presented below extend by analytic continuation to the entire domain U.

a - Derivatives

Lemma (L2): Differentiation of the amplitude

$\forall u \in \mathbb{R}$, $am'(u) = dn(u)$

Proof. (L2): We know that $u = \int_0^{\alpha} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$.
By the fundamental theorem of calculus, differentiating gives:
$\frac{du}{d\alpha} = \frac{1}{\sqrt{1-k^2\sin^2(\alpha)}}$,
so $\frac{d\alpha}{du} = \sqrt{1-k^2\sin^2(\alpha)}$.
Since $\alpha = am(u)$,
we get $am'(u) = dn(u)$.
Hence the result. ■

Proposition (P14): Differentiation

For all $u \in \mathbb{R}$:

$sn'(u) = cn(u)dn(u)$,

$cn'(u) = -sn(u)dn(u)$,

$dn'(u) = -k^2 sn(u)cn(u)$.

Proof. (P14): Let $u \in \mathbb{C}$

We know that $sn(u) = \sin(am(u))$

Therefore, by the chain rule,

$sn'(u) = am'(u)\cos(am(u))$

so $sn'(u) = dn(u)\cos(am(u))$ by the lemma.

Therefore $sn'(u) = dn(u)cn(u)$.

Likewise for cn.

We know that $dn^2(u) + k^2 \cdot sn^2(u) = 1$.

Differentiating: $2dn(u)dn'(u) + k^2 2sn(u)sn'(u) = 0$.

Substituting $sn'(u)$ by $cn(u)dn(u)$ and dividing by $2dn(u)$:

$dn'(u) = -k^2 sn(u)cn(u)$.

Hence the result. ∎

b - Antiderivatives

Proposition (P15): Antiderivatives

Let $C \in \mathbb{R}$.

For all $u \in \mathbb{R}$:

$\int sn(u)du = \frac{1}{k}\ln|dn(u) - kcn(u)| + C$,

$\int cn(u)du = \frac{1}{k}\arccos(dn(u)) + C$,

$\int dn(u)du = am(u) + C$.

Proof. (P15): Let $u \in \mathbb{R}$.

For dn: follows directly from the above.

Set $SN(u) := \frac{1}{k}\ln|dn(u) - kcn(u)| + C$.

We have $SN'(u) = \frac{1}{k}\frac{dn'(u)-ksn'(u)}{dn(u)-kcn(u)} = \frac{1}{k}\frac{-k^2 sn(u)cn(u)-k(-sn(u)dn(u))}{dn(u)-kcn(u)} = \frac{1}{k}\frac{ksn(u)[-kcn(u)+dn(u)]}{dn(u)-kcn(u)} = sn(u)$.

Set $CN(u) := \frac{1}{k}\arccos(dn(u)) + C$.

We have $CN'(u) = \frac{1}{k}(\frac{-1}{\sqrt{1-dn^2(u)}})dn'(u) = \frac{1}{k}(\frac{-1}{\sqrt{k^2sn^2(u)}})[-k^2sn(u)cn(u)] = cn(u)$.
Hence the result. ■

8 - Differential equations

Proposition (P16): Differential equations

The function sn is a solution of the differential equation:
$y'' + (1+k^2)y - 2k^2y^3 = 0$.
The function cn is a solution of the differential equation:
$y'' + (1-2k^2)y + 2k^2y^3 = 0$.
The function dn is a solution of the differential equation:
$y'' - (2-k^2)y + 2y^3 = 0$.

Proof. (P16): Let $u \in \mathbb{R}$.
We know that $sn'(u) = cn(u)dn(u)$,
so $sn''(u) = cn'(u)dn(u) + cn(u)dn'(u)$,
so $sn''(u) = [-sn(u)dn(u)]dn(u) + cn(u)[-k^2sn(u)cn(u)]$,
so $sn''(u) = -sn(u)dn^2(u) - k^2sn(u)cn^2(u)$.
Using the fundamental relations P7 and P8:
$sn''(u) = -sn(u)[1-k^2sn^2(u)] - k^2sn(u)[1-sn^2(u)]$,
so $sn''(u) = -sn(u) + k^2sn^3(u) - k^2sn(u) + k^2sn^3(u)$,
so $sn''(u) = -(1+k^2)sn(u) + 2k^2sn^3(u)$.
We proceed likewise for cn and dn.
Hence the result. ■

III - APPLICATIONS

Preliminary remark:

Having established the rigour necessary for the definition of Jacobi functions, this third section adopts a more applied approach. We will favour here the interpretation of results and the highlighting of links between the various domains addressed, over formalism. In this section, all properties and theorems will be considered as admitted and will not be proved.

1 - Arithmetic-Geometric Mean (AGM)

To numerically evaluate the Jacobi functions and their periods, direct integration is often inefficient. One then uses the Arithmetic-Geometric Mean (AGM) algorithm, a very powerful method.

a - Definition

Definition (D12): Arithmetic-geometric sequences

Let two strictly positive reals a_0 and b_0 such that $a_0 \geq b_0$.
We define the coupled sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ by the following recurrence relations:

$$\forall n \in \mathbb{N}, \begin{cases} a_{n+1} = \frac{a_n+b_n}{2}, & \textit{(Arithmetic mean)} \\ b_{n+1} = \sqrt{a_n b_n}. & \textit{(Geometric mean)} \end{cases}$$

Proposition (P17): Convergence and limit

One can show that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are adjacent. They converge to a unique common limit, denoted $M(a_0, b_0)$, called the arithmetic-geometric mean of a_0 and b_0.

b - Application

Recall:
We defined previously:
$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$.

Theorem (T1): Gauss - Computation of the period

Let $k \in [0, 1[$.
The connection to our study lies in the ability of the AGM to compute the first-kind integral K without resorting to an impossible primitive calculation:
$K = \frac{\pi}{2M(1,k')}$,
where $k' = \sqrt{1-k^2}$.

From a practical standpoint, this tool is fundamental for two reasons:

The circularisation analogy: Each iteration of the AGM reduces the gap between a_n and b_n. Geometrically, this amounts to progressively transforming an ellipse into a perfect circle. The limit $M(1, k')$ represents the radius of the final circle, which explains the presence of π in the formula for the period.
Quadratic convergence: The AGM has so-called "quadratic" convergence: the number of correct significant digits doubles at each step. In just 4 or 5 iterations, one obtains sufficient precision for any physical application, whereas classical methods would require thousands of computations.

2 - Simple pendulum

The most emblematic application of Jacobi functions is undoubtedly the exact solution of the simple heavy pendulum, governed by the equation:
$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0,$
where θ is the angle of oscillation, l the length of the rod, and g the gravitational constant. While the linear approximation $\sin(\theta) \sim \theta$ limits the study to small oscillations, the Jacobi formalism allows the motion to be described for any amplitude.

a - Expression and period

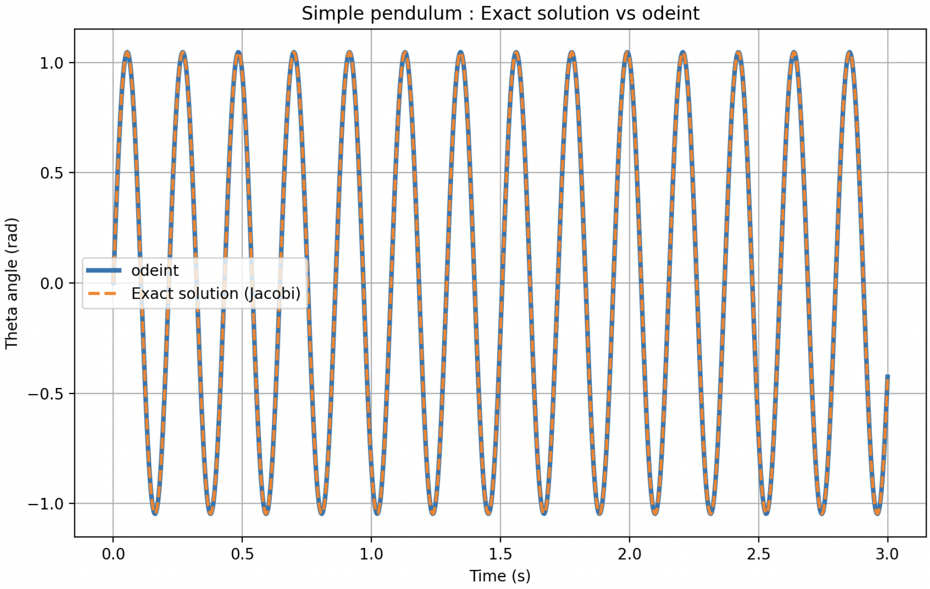
Starting from the conservation of mechanical energy and performing a change of variables, one establishes that the oscillation angle θ is governed by the following relation:
$\theta(t) = 2\arcsin(k \cdot sn(\omega_0 t, k)),$
where $k = \sin(\frac{\theta_0}{2})$ is the Jacobi modulus, and $\omega_0 = \sqrt{\frac{g}{l}}$ is the natural angular frequency.
Using the definition of the period K given in (D12), we can express the real period T of the pendulum:
$T = \frac{4K}{\omega_0}.$
Using the AGM algorithm presented above, this period can be computed with extreme precision.

Remark:
For large angles, T increases significantly: this is the phenomenon of non-isochronism of oscillations.

b - Numerical simulation

To validate this theoretical approach, one can use a Python script comparing the exact solution (Jacobi) with a classical numerical integration (odeint method).

We place ourselves in the case where $\theta_0 = \frac{\pi}{3}$, $l = 10$ cm and $g = 9.81\ m \cdot s^{-2}$.



The graph above confirms the perfect superposition of the curves. The use of Jacobi functions is highly efficient for modelling complex physical systems without accumulating numerical approximation errors.

Note:
The proof of the exact expression for the oscillation angle of the pendulum, along with the Python code, are available at noahroetman.github.io.

3 - Nonlinear waves

If Jacobi functions describe the swinging of a pendulum, they are also at the heart of the theory of solitons: solitary waves that propagate without deforming.
The Korteweg-de Vries equation models waves in shallow water:
$\frac{\partial\phi}{\partial t} + \frac{\partial^3\phi}{\partial x^3} + 6\phi\frac{\partial\phi}{\partial x} = 0.$
One can show that the periodic solution of this equation is not expressed using a sine, but with the Jacobi function cn. This is why such waves are called cnoidal waves (the “cn” coming from the function).
Unlike a classical sinusoidal swell, cnoidal waves have very pronounced crests and very flat troughs. This is exactly what is observed on coastlines when the water depth decreases.
When the modulus k tends to 1, the period becomes infinite. The wave no longer repeats: one obtains a single hump called a soliton. Mathematically, the function cn becomes a hyperbolic secant ($sech = \frac{1}{cosh}$), describing an isolated wave capable of travelling thousands of kilometres without damping.

4 - Cryptography and number theory

As we have seen, Jacobi elliptic functions are related to elliptic curves. These are today used in cryptography. In a continuous setting, they allow the parametrisation of such curves and the expression of their addition formulae, which reveals their group structure.
In cryptography, the same structure appears, but in a discrete setting: one works with elliptic curves defined over the field $\mathbb{Z}/p\mathbb{Z}$.
The Ed25519 protocol is one example. It rests on the fact that it is easy to compute multiples of a point on the curve, but extremely difficult to recover the multiplier from the result (discrete logarithm problem).
The Jacobi functions are therefore not used directly, but they have played an important theoretical role in understanding the properties of elliptic curves upon which this cryptography rests.

CONCLUSION

The aim of this study was to better understand the Jacobi elliptic functions, born from a simple question: measuring the arc of an ellipse, in order to grasp their full mathematical depth and utility. This work shows that they are not mere geometric curiosities. Their fundamental property, double periodicity in the complex plane, governs their behaviour and explains their richness.
Ultimately, the Jacobi functions illustrate well how an idea born from geometry can evolve into a very powerful mathematical tool, capable of providing an exact description of phenomena that physics often treats with approximations.